

DATABASE FOR REAL HYPERFIELDS ALGORITHM

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ABSTRACT. We study finite real hyperfields with the cyclic positive cones. We present an algorithm for determining all such hyperfields up to isomorphism and compute its C-characteristic. By a computer programming we give a complete classification of finite real hyperfields of orders up to 13 and determine which C-characteristics appear in such hyperfields with orders up to 17.

In this database we present the numerical results concerning our algorithm in the case when the positive cone is a cyclic group of order N , for $N \leq 8$.

1. NOTATION

We consider a real hyperfield H with a positive cone P that is a finite cyclic group $\langle a \rangle$. We will use the following notation:

- N denotes the order of the positive cone, i.e., $P = \{1, a, a^2, \dots, a^{N-1}\}$.
- $K = \left\lfloor \frac{N}{2} \right\rfloor$.

Any subset $S = \{a^{i_1}, a^{i_2}, \dots, a^{i_l}\} \subseteq P$ can be identified with the set of exponents

$$I^+ = \{i_1, i_2, \dots, i_l\} \subseteq \{0, 1, 2, \dots, N-1\}.$$

Next, any nonempty subset I^+ of the set $\{0, 1, 2, \dots, N-1\}$ can be uniquely identified with a number $c \in \{1, \dots, 2^N - 1\}$ via the map:

$$(1) \quad I^+ \mapsto c = \sum_{i \in I^+} 2^i.$$

Note that c can be written as a nonzero binary number $(c_{N-1} \dots c_0)_2$ of N bits and the map

$$c = (c_{N-1} \dots c_1 c_0)_2 \mapsto I_c^+ = \{\ell \mid c_\ell = 1\}$$

is the inverse mapping to (1).

Each real finite hyperfield with a cyclic positive cone is uniquely determined by a sequence of nonzero subsets $(1 + a^0, 1 + a^1, \dots, 1 + a^K)$ where $1 + a^i \subset P$. It can be identified with a $(K+1)$ -tuple of the form $\underline{s} = (s_0, s_1, \dots, s_K)$, where the number $s_l \in \{1, 2, \dots, 2^N - 1\}$ is equal to the index of the subset $1 + a^l$.

2. THE CASE $N = 1$

If $N = 1$, then there is only one real hyperfield \mathbb{S} of C-characteristic equal to 1, given by 1-tuple (1). It is called hyperfield of signs. The hyperfield $\mathbb{S} := \{-1, 0, 1\}$ with the usual multiplication and the hyperaddition $+$ defined as follows:

+	-1	0	1
-1	$\{-1\}$	$\{-1\}$	$\{-1, 0, 1\}$
0	$\{-1\}$	$\{0\}$	$\{1\}$
1	$\{-1, 0, 1\}$	$\{1\}$	$\{1\}$

TABLE 1. The table of hyperaddition for the sign hyperfield \mathbb{S} .

3. THE CASE $N = 2$

If $N = 2$, then there are 2 non-isomorphic real hyperfields give by 2-tubes $(1, 3)$ and $(3, 3)$. Each hyperfield has C-characteristic equal to 1.

$+$	$-a$	-1	0	1	a
$-a$	$\{-a\}$	$-P$	$\{-a\}$	$\{-a, 1\}$	H
-1	$-P$	$\{-1\}$	$\{-1\}$	H	$\{-1, a\}$
0	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$
1	$\{-a, 1\}$	H	$\{1\}$	$\{1\}$	P
a	H	$\{-1, a\}$	$\{a\}$	P	$\{a\}$

TABLE 2. The table of hyperaddition for the tube $(1, 3)$.

$+$	$-a$	-1	0	1	a
$-a$	$-P$	$-P$	$\{-a\}$	H^*	H
-1	$-P$	$-P$	$\{-1\}$	H	H^*
0	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$
1	H^*	H	$\{1\}$	P	P
a	H	H^*	$\{a\}$	P	P

TABLE 3. The table of hyperaddition for the tube $(3, 3)$.4. THE CASE $N = 3$

If $N = 3$ the hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1)$. We have checked exactly 49 cases, of which 11 are hyperfields. This is 22.449% of all cases, while non-isomorphic hyperfields are 8. This is 16,3265% of the total. The following list includes all non-isomorphic real hyperfields of order 7 with the cyclic positive cone of order $N = 3$.

Hyperfields of C-characteristic equal to 1:

$(1, 3), (7, 3), (3, 5), (7, 5), (3, 7), (7, 7)$

Hyperfields of C-characteristic equal to 2:

$(6, 3), (6, 7)$

Let us denote by $[\underline{s}]$ a isomorphisms class of real hyperfield $\mathbb{H}(\underline{s})$. There are three isomorphism classes of order two:

Isomorphism classes of order two:

$$[(3, 5)] = \{(3, 5), (5, 6)\} \quad [(7, 5)] = \{(7, 5), (7, 6)\} \quad [(3, 7)] = \{(3, 7), (5, 7)\}$$

the other isomorphism classes are singletons.

$+$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-a^2\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{-a^2, 1\}$	$\{-a^2, a\}$	H
$-a$	$\{-a^2, -a\}$	$\{-a\}$	$\{-a, -1\}$	$\{-a\}$	$\{-a, 1\}$	H	$\{-a, a^2\}$
-1	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-1\}$	$\{-1\}$	H	$\{-1, a\}$	$\{-1, a^2\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$\{-a^2, 1\}$	$\{-a, 1\}$	H	$\{1\}$	$\{1\}$	$\{1, a\}$	$\{1, a^2\}$
a	$\{-a^2, a\}$	H	$\{-1, a\}$	$\{a\}$	$\{1, a\}$	$\{a\}$	$\{a, a^2\}$
a^2	H	$\{-a, a^2\}$	$\{-1, a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{a^2\}$

 TABLE 4. The table of hyperaddition for the tube $(1, 3)$.

$+$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$-P$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$	H
$-a$	$\{-a^2, -a\}$	$-P$	$\{-a, -1\}$	$\{-a\}$	$\{\pm 1, \pm a\}$	H	$\{\pm a, \pm a^2\}$
-1	$\{-a^2, -1\}$	$\{-a, -1\}$	$-P$	$\{-1\}$	H	$\{\pm 1, \pm a\}$	$\{\pm 1, \pm a^2\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	H	$\{1\}$	P	$\{1, a\}$	$\{1, a^2\}$
a	$\{\pm a, \pm a^2\}$	H	$\{\pm 1, \pm a\}$	$\{a\}$	$\{1, a\}$	P	$\{a, a^2\}$
a^2	H	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	P

 TABLE 5. The table of hyperaddition for the tube $(7, 3)$.

$+$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-a^2, -a\}$	$\{-a^2\}$	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a\}$	$\{0, \pm 1, \pm a^2\}$
$-a$	$\{-a, -1\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a\}$	$\{\pm 1, \pm a^2\}$	$\{0, \pm a, \pm a^2\}$	$\{\pm 1, \pm a\}$
-1	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-1\}$	$\{0, \pm 1, \pm a\}$	$\{\pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{0, \pm 1, \pm a\}$	$\{1\}$	$\{1, a\}$	$\{1, a^2\}$	$\{a, a^2\}$
a	$\{\pm 1, \pm a\}$	$\{0, \pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{1, a\}$
a^2	$\{0, \pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	$\{\pm a, \pm a^2\}$	$\{a^2\}$	$\{a, a^2\}$	$\{1, a\}$	$\{1, a^2\}$

 TABLE 6. The table of hyperaddition for the tube $(3, 5)$.

$+$	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$-P$	$\{-a, -1\}$	$\{-a^2, -a\}$	$\{-a^2\}$	$H^* - \{1\}$	$H^* - \{-a^2\}$	$\{0, \pm 1, \pm a^2\}$
$-a$	$\{-a, -1\}$	$-P$	$\{-a^2, -1\}$	$\{-a\}$	$H^* - \{-a\}$	$\{0, \pm a, \pm a^2\}$	$H^* - \{a^2\}$
-1	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$-P$	$\{-1\}$	$\{0, \pm 1, \pm a\}$	$H^* - \{a\}$	$H^* - \{-1\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$H^* - \{1\}$	$H^* - \{-a\}$	$\{0, \pm 1, \pm a\}$	$\{1\}$	P	$\{1, a^2\}$	$\{a, a^2\}$
a	$H^* - \{-a^2\}$	$\{0, \pm a, \pm a^2\}$	$H^* - \{a\}$	$\{a\}$	$\{1, a^2\}$	P	$\{1, a\}$
a^2	$\{0, \pm 1, \pm a^2\}$	$H^* - \{a^2\}$	$H^* - \{-1\}$	$\{a^2\}$	$\{a, a^2\}$	$\{1, a\}$	P

 TABLE 7. The table of hyperaddition for the tube $(7, 5)$.

+	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-a^2, -1\}$	$-P$	$-P$	$\{-a^2\}$	$H^* - \{-1\}$	$H^* - \{a^2\}$	H
$-a$	$-P$	$\{-a^2, -a\}$	$-P$	$\{-a\}$	$H^* - \{a\}$	H	$H^* - \{-a^2\}$
-1	$-P$	$-P$	$\{-a, -1\}$	$\{-1\}$	H	$H^* - \{-a\}$	$H^* - \{1\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$H^* - \{-1\}$	$H^* - \{a\}$	H	$\{1\}$	$\{1, a\}$	P	P
a	$H^* - \{a^2\}$	H	$H^* - \{-a\}$	$\{a\}$	P	$\{a, a^2\}$	P
a^2	H	$H^* - \{-a^2\}$	$H^* - \{1\}$	$\{a^2\}$	P	P	$\{1, a^2\}$

TABLE 8. The table of hyperaddition for the tube $(3, 7)$.

+	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$-P$	$-P$	$-P$	$\{-a^2\}$	H^*	H^*	H
$-a$	$-P$	$-P$	$-P$	$\{-a\}$	H^*	H	H^*
-1	$-P$	$-P$	$-P$	$\{-1\}$	H	H^*	H^*
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	H^*	H^*	H	$\{1\}$	P	P	P
a	H^*	H	H^*	$\{a\}$	P	P	P
a^2	H	H^*	H^*	$\{a^2\}$	P	P	P

TABLE 9. The table of hyperaddition for the tube $(7, 7)$.

+	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-a, -1\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$	$\{0, \pm 1, \pm a\}$
$-a$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-a\}$	$\{\pm 1, \pm a\}$	$\{0, \pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$
-1	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-a^2, -a\}$	$\{-1\}$	$\{0, \pm a, \pm a^2\}$	$\{\pm 1, \pm a\}$	$\{\pm 1, \pm a^2\}$
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	$\{0, \pm a, \pm a^2\}$	$\{1\}$	$\{a, a^2\}$	$\{1, a\}$	$\{1, a^2\}$
a	$\{\pm a, \pm a^2\}$	$\{0, \pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	$\{a\}$	$\{1, a\}$	$\{1, a^2\}$	$\{a, a^2\}$
a^2	$\{0, \pm 1, \pm a\}$	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{1, a\}$

TABLE 10. The table of hyperaddition for the tube $(6, 3)$.

+	$-a^2$	$-a$	-1	0	1	a	a^2
$-a^2$	$\{-a, -1\}$	$-P$	$-P$	$\{-a^2\}$	H^*	H^*	$\{0, \pm 1, \pm a\}$
$-a$	$-P$	$\{-a^2, -1\}$	$-P$	$\{-a\}$	H^*	$\{0, \pm 1, \pm a^2\}$	H^*
-1	$-P$	$-P$	$\{-a^2, -a\}$	$\{-1\}$	$\{0, \pm a, \pm a^2\}$	H^*	H^*
0	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$
1	H^*	H^*	$\{0, \pm a, \pm a^2\}$	$\{1\}$	$\{a, a^2\}$	P	P
a	H^*	$\{0, \pm 1, \pm a^2\}$	H^*	$\{a\}$	P	$\{1, a^2\}$	P
a^2	$\{0, \pm 1, \pm a\}$	H^*	H^*	$\{a^2\}$	P	P	$\{1, a\}$

TABLE 11. The table of hyperaddition for the tube $(6, 7)$.

5. THE CASE $N = 4$

The hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1, s_2)$. We have checked exactly 3375 cases, of which 30 are hyperfields. This is 0.888889% of all cases, while non-isomorphic hyperfields are 20. This is 0.592593% of the total. The following list includes all non-isomorphic real hyperfields of order 9 with the cyclic positive cone of order $N = 4$.

Hyperfields of C-characteristic equal to 1:

(1, 3, 5), (5, 3, 5), (15, 3, 5), (5, 15, 5), (1, 3, 15), (5, 3, 15), (7, 7, 15), (9, 7, 15), (13, 7, 15), (15, 7, 15), (7, 13, 15), (11, 13, 15), (13, 13, 15), (15, 13, 15), (7, 15, 15), (11, 15, 15), (15, 15, 15)

Hyperfields of C-characteristic equal to 2:

(14, 3, 5), (14, 7, 15), (14, 15, 15)

There are 10 isomorphism classes of order two:

Isomorphism classes of order two:

$$\begin{aligned}
 [(7, 7, 15)] &= \{(7, 7, 15), (13, 11, 15)\} & [(11, 13, 15)] &= \{(11, 13, 15), (11, 14, 15)\} \\
 [(9, 7, 15)] &= \{(9, 7, 15), (3, 11, 15)\} & [(13, 13, 15)] &= \{(13, 13, 15), (7, 14, 15)\} \\
 [(13, 7, 15)] &= \{(13, 7, 15), (7, 11, 15)\} & [(15, 13, 15)] &= \{(15, 13, 15), (15, 14, 15)\} \\
 [(15, 7, 15)] &= \{(15, 7, 15), (15, 11, 15)\} & [(7, 15, 15)] &= \{(7, 15, 15), (13, 15, 15)\} \\
 [(7, 13, 15)] &= \{(7, 13, 15), (13, 14, 15)\} & [(14, 7, 15)] &= \{(14, 7, 15), (14, 11, 15)\}
 \end{aligned}$$

the other isomorphisms classes are singletons.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3\}$	$\{-a^3, -a^2\}$	$\{-a^3, -a\}$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{-a^3, 1\}$	$\{-a^3, a\}$	$\{-a^3, a^2\}$	H
$-a^2$	$\{-a^3, -a^2\}$	$\{-a^2\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{-a^2, 1\}$	$\{-a^2, a\}$	H	$\{-a^2, a^3\}$
$-a$	$\{-a^3, -a\}$	$\{-a^2, -a\}$	$\{-a\}$	$\{-a, -1\}$	$\{-a\}$	$\{-a, 1\}$	H	$\{-a, a^2\}$	$\{-a, a^3\}$
-1	$\{-a^3, -1\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-1\}$	$\{-1\}$	H	$\{-1, a\}$	$\{-1, a^2\}$	$\{-1, a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{-a^3, 1\}$	$\{-a^2, 1\}$	$\{-a, 1\}$	H	$\{1\}$	$\{1\}$	$\{1, a\}$	$\{1, a^2\}$	$\{1, a^3\}$
a	$\{-a^3, a\}$	$\{-a^2, a\}$	H	$\{-1, a\}$	$\{a\}$	$\{1, a\}$	$\{a\}$	$\{a, a^2\}$	$\{a, a^3\}$
a^2	$\{-a^3, a^2\}$	H	$\{-a, a^2\}$	$\{-1, a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{a^2\}$	$\{a^2, a^3\}$
a^3	H	$\{-a^2, a^3\}$	$\{-a, a^3\}$	$\{-1, a^3\}$	$\{a^3\}$	$\{1, a^3\}$	$\{a, a^3\}$	$\{a^2, a^3\}$	$\{a^3\}$

TABLE 12. The table of hyperaddition for the tube (1, 3, 5).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a\}$	$\{-a^3, -a^2\}$	$\{-a^3, -a\}$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{-a^3, 1\}$	$\{\pm a, \pm a^3\}$	$\{-a^3, a^2\}$	H
$-a^2$	$\{-a^3, -a^2\}$	$\{-a^2, -1\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{-a^2, a\}$	H	$\{-a^2, a^3\}$
$-a$	$\{-a^3, -a\}$	$\{-a^2, -a\}$	$\{-a^3, -a\}$	$\{-a, -1\}$	$\{-a\}$	$\{-a, 1\}$	H	$\{-a, a^2\}$	$\{\pm a, \pm a^3\}$
-1	$\{-a^3, -1\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-a^2, -1\}$	$\{-1\}$	H	$\{-1, a\}$	$\{\pm 1, \pm a^2\}$	$\{-1, a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{-a^3, 1\}$	$\{\pm 1, \pm a^2\}$	$\{-a, 1\}$	H	$\{1\}$	$\{1, a^2\}$	$\{1, a\}$	$\{1, a^2\}$	$\{1, a^3\}$
a	$\{\pm a, \pm a^3\}$	$\{-a^2, a\}$	H	$\{-1, a\}$	$\{a\}$	$\{1, a\}$	$\{a, a^3\}$	$\{a, a^2\}$	$\{a, a^3\}$
a^2	$\{-a^3, a^2\}$	H	$\{-a, a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{1, a^2\}$	$\{a^2, a^3\}$
a^3	H	$\{-a^2, a^3\}$	$\{\pm a, \pm a^3\}$	$\{-1, a^3\}$	$\{a^3\}$	$\{1, a^3\}$	$\{a, a^3\}$	$\{a^2, a^3\}$	$\{a, a^3\}$

TABLE 13. The table of hyperaddition for the tube (5, 3, 5).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$-P$	$\{-a^3, -a^2\}$	$\{-a^3, -a\}$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{\pm 1, \pm a^3\}$	$\{\pm a, \pm a^3\}$	$\{\pm a^2, \pm a^3\}$	H
$-a^2$	$\{-a^3, -a^2\}$	$-P$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$	H	$\{\pm a^2, \pm a^3\}$
$-a$	$\{-a^3, -a\}$	$\{-a^2, -a\}$	$-P$	$\{-a, -1\}$	$\{-a\}$	$\{\pm 1, \pm a\}$	H	$\{\pm a, \pm a^2\}$	$\{\pm a, \pm a^3\}$
-1	$\{-a^3, -1\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$-P$	$\{-1\}$	H	$\{\pm 1, \pm a\}$	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{\pm 1, \pm a^3\}$	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	H	$\{1\}$	P	$\{1, a\}$	$\{1, a^2\}$	$\{1, a^3\}$
a	$\{\pm a, \pm a^3\}$	$\{\pm a, \pm a^2\}$	H	$\{\pm 1, \pm a\}$	$\{a\}$	$\{1, a\}$	P	$\{a, a^2\}$	$\{a, a^3\}$
a^2	$\{\pm a^2, \pm a^3\}$	H	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	P	$\{a^2, a^3\}$
a^3	H	$\{\pm a^2, \pm a^3\}$	$\{\pm a, \pm a^3\}$	$\{\pm 1, \pm a^3\}$	$\{a^3\}$	$\{1, a^3\}$	$\{a, a^3\}$	$\{a^2, a^3\}$	P

TABLE 14. The table of hyperaddition for the tube $(15, 3, 5)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a\}$	$-P$	$\{-a^3, -a\}$	$-P$	$\{-a^3\}$	$\{-a, -a^3, 1, a^2\}$	H^*	$\{-a, -a^3, 1, a^2\}$	H
$-a^2$	$-P$	$\{-a^2, -1\}$	$-P$	$\{-a^2, -1\}$	$\{-a^2\}$	H^*	$\{-1, -a^2, a, a^3\}$	H	$\{-1, -a^2, a, a^3\}$
$-a$	$\{-a^3, -a\}$	$-P$	$\{-a^3, -a\}$	$-P$	$\{-a\}$	$\{-a, -a^3, 1, a^2\}$	H	$\{-a, -a^3, 1, a^2\}$	H^*
-1	$-P$	$\{-a^2, -1\}$	$-P$	$\{-a^2, -1\}$	$\{-1\}$	H	$\{-1, -a^2, a, a^3\}$	H^*	$\{-1, -a^2, a, a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{-a, -a^3, 1, a^2\}$	H^*	$\{-a, -a^3, 1, a^2\}$	H	$\{1\}$	$\{1, a^2\}$	P	$\{1, a^2\}$	P
a	H^*	$\{-1, -a^2, a, a^3\}$	H	$\{-1, -a^2, a, a^3\}$	$\{a\}$	P	$\{a, a^3\}$	P	$\{a, a^3\}$
a^2	$\{-a, -a^3, 1, a^2\}$	H	$\{-a, -a^3, 1, a^2\}$	H^*	$\{a^2\}$	$\{1, a^2\}$	P	$\{1, a^2\}$	P
a^3	H	$\{-1, -a^2, a, a^3\}$	H^*	$\{-1, -a^2, a, a^3\}$	$\{a^3\}$	P	$\{a, a^3\}$	P	$\{a, a^3\}$

TABLE 15. The table of hyperaddition for the tube $(5, 15, 5)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3\}$	$\{-a^3, -a^2\}$	$-P$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{-a^2, -a^3, 1, a\}$	$\{-a^3, a\}$	$\{-1, -a^3, a, a^2\}$	H
$-a^2$	$\{-a^3, -a^2\}$	$\{-a^2\}$	$\{-a^2, -a\}$	$-P$	$\{-a^2\}$	$\{-a^2, 1\}$	$\{-a^2, -a^3, 1, a\}$	H	$\{-a, -a^2, 1, a^3\}$
$-a$	$-P$	$\{-a^2, -a\}$	$\{-a\}$	$\{-a, -1\}$	$\{-a\}$	$\{-a, -a^2, 1, a^3\}$	H	$\{-1, -a, a^2, a^3\}$	$\{-a, a^3\}$
-1	$\{-a^3, -1\}$	$-P$	$\{-a, -1\}$	$\{-1\}$	$\{-1\}$	H	$\{-1, -a^3, a, a^2\}$	$\{-1, a^2\}$	$\{-1, -a, a^2, a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{-a^2, -a^3, 1, a\}$	$\{-a^2, 1\}$	$\{-a, -a^2, 1, a^3\}$	H	$\{1\}$	$\{1\}$	$\{1, a\}$	P	$\{1, a^3\}$
a	$\{-a^3, a\}$	$\{-a^2, -a^3, 1, a\}$	H	$\{-1, -a^3, a, a^2\}$	$\{a\}$	$\{1, a\}$	$\{a\}$	$\{a, a^2\}$	P
a^2	$\{-1, -a^3, a, a^2\}$	H	$\{-1, -a, a^2, a^3\}$	$\{-1, a^2\}$	$\{a^2\}$	P	$\{a, a^2\}$	$\{a^2\}$	$\{a^2, a^3\}$
a^3	H	$\{-a, -a^2, 1, a^3\}$	$\{-a, a^3\}$	$\{-1, -a, a^2, a^3\}$	$\{a^3\}$	$\{1, a^3\}$	P	$\{a^2, a^3\}$	$\{a^3\}$

TABLE 16. The table of hyperaddition for the tube $(1, 3, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a\}$	$\{-a^3, -a^2\}$	$-P$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{-a^2, -a^3, 1, a\}$	$\{\pm a, \pm a^3\}$	$\{-1, -a^3, a, a^2\}$	H
$-a^2$	$\{-a^3, -a^2\}$	$\{-a^2, -1\}$	$\{-a^2, -a\}$	$-P$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{-a^2, -a^3, 1, a\}$	H	$\{-a, -a^2, 1, a^3\}$
$-a$	$-P$	$\{-a^2, -a\}$	$\{-a^3, -a\}$	$\{-a, -1\}$	$\{-a\}$	$\{-a, -a^2, 1, a^3\}$	H	$\{-1, -a, a^2, a^3\}$	$\{\pm a, \pm a^3\}$
-1	$\{-a^3, -1\}$	$-P$	$\{-a, -1\}$	$\{-a^2, -1\}$	$\{-1\}$	H	$\{-1, -a^3, a, a^2\}$	$\{\pm 1, \pm a^2\}$	$\{-1, -a, a^2, a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{-a^2, -a^3, 1, a\}$	$\{\pm 1, \pm a^2\}$	$\{-a, -a^2, 1, a^3\}$	H	$\{1\}$	$\{1, a^2\}$	$\{1, a\}$	P	$\{1, a^3\}$
a	$\{\pm a, \pm a^3\}$	$\{-a^2, -a^3, 1, a\}$	H	$\{-1, -a^3, a, a^2\}$	$\{a\}$	$\{1, a\}$	$\{a, a^3\}$	$\{a, a^2\}$	P
a^2	$\{-1, -a^3, a, a^2\}$	H	$\{-1, -a, a^2, a^3\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	P	$\{a, a^2\}$	$\{1, a^2\}$	$\{a^2, a^3\}$
a^3	H	$\{-a, -a^2, 1, a^3\}$	$\{\pm a, \pm a^3\}$	$\{-1, -a, a^2, a^3\}$	$\{a^3\}$	$\{1, a^3\}$	P	$\{a^2, a^3\}$	$\{a, a^3\}$

TABLE 17. The table of hyperaddition for the tube $(5, 3, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3\}$	$H^* - \{-1, -a\}$	$H^* - \{-1, a^2\}$	$H^* - \{1, a^3\}$	H
$-a^2$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2\}$	$H^* - \{-a^3, a\}$	$H^* - \{a^2, a^3\}$	H	$H^* - \{-1, -a^3\}$
$-a$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a^2, -a\}$	$\{-a^2, -a, -1\}$	$\{-a\}$	$H^* - \{a, a^2\}$	H	$H^* - \{-a^2, -a^3\}$	$H^* - \{-a^2, 1\}$
-1	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2, -a, -1\}$	$\{-a^2, -a, -1\}$	$\{-1\}$	H	$H^* - \{-a, -a^2\}$	$H^* - \{-a, a^3\}$	$H^* - \{1, a\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-1, -a\}$	$H^* - \{-a^3, a\}$	$H^* - \{a, a^2\}$	H	$\{1\}$	$\{1, a, a^2\}$	$\{1, a, a^2\}$	P	$\{1, a, a^3\}$
a	$H^* - \{-1, a^2\}$	$H^* - \{a^2, a^3\}$	H	$H^* - \{-a, -a^2\}$	$\{a\}$	$\{1, a, a^2\}$	$\{a, a^2, a^3\}$	$\{a, a^2, a^3\}$	P
a^2	$H^* - \{1, a^3\}$	H	$H^* - \{-a^2, -a^3\}$	$H^* - \{-a, a^3\}$	$\{a^2\}$	P	$\{a, a^2, a^3\}$	$\{1, a^2, a^3\}$	$\{1, a^2, a^3\}$
a^3	H	$H^* - \{-1, -a^3\}$	$H^* - \{-a^2, 1\}$	$H^* - \{1, a\}$	$\{a^3\}$	$\{1, a, a^3\}$	P	$\{1, a^2, a^3\}$	$\{1, a, a^3\}$

TABLE 18. The table of hyperaddition for the tube $(7, 7, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a^2\}$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3\}$	$H^* - \{-a, a^3\}$	$\{-a^2, -a^3, 1, a\}$	$H^* - \{-a^2, 1\}$	H
$-a^2$	$\{-a^3, -a^2, -1\}$	$\{-a^2, -a\}$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2\}$	$\{-a, -a^2, 1, a^3\}$	$H^* - \{-a, a^3\}$	H	$H^* - \{-1, a^2\}$
$-a$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a, -1\}$	$\{-a^2, -a, -1\}$	$\{-a\}$	$H^* - \{-1, a^2\}$	H	$H^* - \{-a^3, a\}$	$\{-1, -a, a^2, a^3\}$
-1	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2, -a, -1\}$	$\{-a^3, -1\}$	$\{-1\}$	H	$H^* - \{-a^2, 1\}$	$\{-1, -a^3, a, a^2\}$	$H^* - \{-a^3, a\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-a, a^3\}$	$\{-a, -a^2, 1, a^3\}$	$H^* - \{-1, a^2\}$	H	$\{1\}$	$\{1, a^3\}$	$\{1, a, a^2\}$	P	$\{1, a, a^3\}$
a	$\{-a^2, -a^3, 1, a\}$	$H^* - \{-a, a^3\}$	H	$H^* - \{-a^2, 1\}$	$\{a\}$	$\{1, a, a^2\}$	$\{1, a\}$	$\{a, a^2, a^3\}$	P
a^2	$H^* - \{-a^2, 1\}$	H	$H^* - \{-a^3, a\}$	$\{-1, -a^3, a, a^2\}$	$\{a^2\}$	P	$\{a, a^2, a^3\}$	$\{a, a^2\}$	$\{1, a^2, a^3\}$
a^3	H	$H^* - \{-1, a^2\}$	$\{-1, -a, a^2, a^3\}$	$H^* - \{-a^3, a\}$	$\{a^3\}$	$\{1, a, a^3\}$	P	$\{1, a^2, a^3\}$	$\{a^2, a^3\}$

TABLE 19. The table of hyperaddition for the tube (9, 7, 15).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3\}$	$H^* - \{-a, a^3\}$	$H^* - \{-1, a^2\}$	$H^* - \{-a^2, 1\}$	H
$-a^2$	$\{-a^3, -a^2, -1\}$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2\}$	$H^* - \{-a^3, a\}$	$H^* - \{-a, a^3\}$	H	$H^* - \{-1, a^2\}$
$-a$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a, -1\}$	$\{-a^2, -a, -1\}$	$\{-a\}$	$H^* - \{-1, a^2\}$	H	$H^* - \{-a^3, a\}$	$H^* - \{-a^2, 1\}$
-1	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-1\}$	H	$H^* - \{-a^2, 1\}$	$H^* - \{-a, a^3\}$	$H^* - \{-a^3, a\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-a, a^3\}$	$H^* - \{-a^3, a\}$	$H^* - \{-1, a^2\}$	H	$\{1\}$	$\{1, a^2, a^3\}$	$\{1, a, a^2\}$	P	$\{1, a, a^3\}$
a	$H^* - \{-1, a^2\}$	$H^* - \{-a, a^3\}$	H	$H^* - \{-a^2, 1\}$	$\{a\}$	$\{1, a, a^2\}$	$\{1, a, a^3\}$	$\{a, a^2, a^3\}$	P
a^2	$H^* - \{-a^2, 1\}$	H	$H^* - \{-a^3, a\}$	$H^* - \{-a, a^3\}$	$\{a^2\}$	P	$\{a, a^2, a^3\}$	$\{1, a, a^2\}$	$\{1, a^2, a^3\}$
a^3	H	$H^* - \{-1, a^2\}$	$H^* - \{-a^2, 1\}$	$H^* - \{-a^3, a\}$	$\{a^3\}$	$\{1, a, a^3\}$	P	$\{1, a^2, a^3\}$	$\{a, a^2, a^3\}$

TABLE 20. The table of hyperaddition for the tube (13, 7, 15).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$-P$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3\}$	$H^* - \{-a\}$	$H^* - \{-1, a^2\}$	$H^* - \{1\}$	H
$-a^2$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2\}$	$H^* - \{-a^3, a\}$	$H^* - \{a^3\}$	H	$H^* - \{-1\}$
$-a$	$-P$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2, -a, -1\}$	$\{-a\}$	$H^* - \{a^2\}$	H	$H^* - \{-a^3\}$	$H^* - \{-a^2, 1\}$
-1	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2, -a, -1\}$	$-P$	$\{-1\}$	H	$H^* - \{-a^2\}$	$H^* - \{-a, a^3\}$	$H^* - \{a\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-a\}$	$H^* - \{-a^3, a\}$	$H^* - \{a^2\}$	H	$\{1\}$	P	$\{1, a, a^2\}$	P	$\{1, a, a^3\}$
a	$H^* - \{-1, a^2\}$	$H^* - \{a^3\}$	H	$H^* - \{-a^2\}$	$\{a\}$	$\{1, a, a^2\}$	P	$\{a, a^2, a^3\}$	P
a^2	$H^* - \{1\}$	H	$H^* - \{-a^3\}$	$H^* - \{-a, a^3\}$	$\{a^2\}$	P	$\{a, a^2, a^3\}$	P	$\{1, a^2, a^3\}$
a^3	H	$H^* - \{-1\}$	$H^* - \{-a^2, 1\}$	$H^* - \{a\}$	$\{a^3\}$	$\{1, a, a^3\}$	P	$\{1, a^2, a^3\}$	P

TABLE 21. The table of hyperaddition for the tube (15, 7, 15).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a, -1\}$	$\{-a^2, -a, -1\}$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3\}$	$H^* - \{\pm 1\}$	H^*	$H^* - \{\pm a^3\}$	$H - \{\pm a^2\}$
$-a^2$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2\}$	H^*	$H^* - \{\pm a^2\}$	$H - \{\pm a\}$	$H^* - \{\pm a^3\}$
$-a$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a^2, -1\}$	$\{-a\}$	$H^* - \{\pm a\}$	$H - \{\pm 1\}$	$H^* - \{\pm a^2\}$	H^*
-1	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^3, -a^2, -1\}$	$\{-a^2, -a, -1\}$	$\{-1\}$	$H - \{\pm a^3\}$	$H^* - \{\pm a\}$	H^*	$H^* - \{\pm 1\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{\pm 1\}$	H^*	$H^* - \{\pm a\}$	$H - \{\pm a^3\}$	$\{1\}$	$\{1, a, a^2\}$	$\{1, a^2, a^3\}$	P	$\{a, a^2, a^3\}$
a	H^*	$H^* - \{\pm a^2\}$	$H - \{\pm 1\}$	$H^* - \{\pm a\}$	$\{a\}$	$\{1, a^2, a^3\}$	$\{a, a^2, a^3\}$	$\{1, a, a^3\}$	P
a^2	$H^* - \{\pm a^3\}$	$H - \{\pm a\}$	$H^* - \{\pm a^2\}$	H^*	$\{a^2\}$	P	$\{1, a, a^3\}$	$\{1, a^2, a^3\}$	$\{1, a, a^2\}$
a^3	$H - \{\pm a^2\}$	$H^* - \{\pm a^3\}$	H^*	$H^* - \{\pm 1\}$	$\{a^3\}$	$\{a, a^2, a^3\}$	P	$\{1, a, a^2\}$	$\{1, a, a^3\}$

TABLE 22. The table of hyperaddition for the tube (7, 13, 15).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a^2, -1\}$	$\{-a^2, -a, -1\}$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3\}$	$H^* - \{1\}$	$H^* - \{-a, a^3\}$	$H^* - \{-a^3\}$	$H - \{\pm a^2\}$
$-a^2$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2\}$	$H^* - \{-1, a^2\}$	$H^* - \{-a^2\}$	$H - \{\pm a\}$	$H^* - \{a^3\}$
$-a$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-a\}$	$H^* - \{-a\}$	$H - \{\pm 1\}$	$H^* - \{a^2\}$	$H^* - \{-a^3, a\}$
-1	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a, -1\}$	$\{-1\}$	$H - \{\pm a^3\}$	$H^* - \{a\}$	$H^* - \{-a^2, 1\}$	$H^* - \{-1\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{1\}$	$H^* - \{-1, a^2\}$	$H^* - \{-a\}$	$H - \{\pm a^3\}$	$\{1\}$	$\{1, a, a^3\}$	$\{1, a^2, a^3\}$	P	$\{a, a^2, a^3\}$
a	$H^* - \{-a, a^3\}$	$H^* - \{-a^2\}$	$H - \{\pm 1\}$	$H^* - \{a\}$	$\{a\}$	$\{1, a^2, a^3\}$	$\{1, a, a^2\}$	$\{1, a, a^3\}$	P
a^2	$H^* - \{-a^3\}$	$H - \{\pm a\}$	$H^* - \{a^2\}$	$H^* - \{-a^2, 1\}$	$\{a^2\}$	P	$\{1, a, a^3\}$	$\{a, a^2, a^3\}$	$\{1, a, a^2\}$
a^3	$H - \{\pm a^2\}$	$H^* - \{a^3\}$	$H^* - \{-a^3, a\}$	$H^* - \{-1\}$	$\{a^3\}$	$\{a, a^2, a^3\}$	P	$\{1, a, a^2\}$	$\{1, a^2, a^3\}$

TABLE 23. The table of hyperaddition for the tube (11, 13, 15).

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a^2, -a\}$	$\{-a^2, -a, -1\}$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3\}$	$H^* - \{1, a^3\}$	H^*	$H^* - \{-a^2, -a^3\}$	$H - \{\pm a^2\}$
$-a^2$	$\{-a^2, -a, -1\}$	$\{-a^2, -a, -1\}$	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2\}$	H^*	$H^* - \{-a, -a^2\}$	$H - \{\pm a\}$	$H^* - \{a^2, a^3\}$
$-a$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-a\}$	$H^* - \{-1, -a\}$	$H - \{\pm 1\}$	$H^* - \{a, a^2\}$	H^*
-1	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a^2, -1\}$	$\{-1\}$	$H - \{\pm a^3\}$	$H^* - \{1, a\}$	H^*	$H^* - \{-1, -a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{1, a^3\}$	H^*	$H^* - \{-1, -a\}$	$H - \{\pm a^3\}$	$\{1\}$	$\{1, a^2, a^3\}$	$\{1, a^2, a^3\}$	P	$\{a, a^2, a^3\}$
a	H^*	$H^* - \{-a, -a^2\}$	$H - \{\pm 1\}$	$H^* - \{1, a\}$	$\{a\}$	$\{1, a^2, a^3\}$	$\{1, a, a^3\}$	$\{1, a, a^3\}$	P
a^2	$H^* - \{-a^2, -a^3\}$	$H - \{\pm a\}$	$H^* - \{a, a^2\}$	H^*	$\{a^2\}$	P	$\{1, a, a^3\}$	$\{1, a, a^2\}$	$\{1, a, a^2\}$
a^3	$H - \{\pm a^2\}$	$H^* - \{a^2, a^3\}$	H^*	$H^* - \{-1, -a^3\}$	$\{a^3\}$	$\{a, a^2, a^3\}$	P	$\{1, a, a^2\}$	$\{a, a^2, a^3\}$

TABLE 24. The table of hyperaddition for the tube $(13, 13, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$-P$	$\{-a^2, -a, -1\}$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3\}$	$H^* - \{1\}$	H^*	$H^* - \{-a^3\}$	$H - \{\pm a^2\}$
$-a^2$	$\{-a^2, -a, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2\}$	H^*	$H^* - \{-a^2\}$	$H - \{\pm a\}$	$H^* - \{a^3\}$
$-a$	$-P$	$\{-a^3, -a, -1\}$	$-P$	$\{-a^3, -a^2, -1\}$	$\{-a\}$	$H^* - \{-a\}$	$H - \{\pm 1\}$	$H^* - \{a^2\}$	H^*
-1	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-1\}$	$H - \{\pm a^3\}$	$H^* - \{a\}$	H^*	$H^* - \{-1\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{1\}$	H^*	$H^* - \{-a\}$	$H - \{\pm a^3\}$	$\{1\}$	P	$\{1, a^2, a^3\}$	P	$\{a, a^2, a^3\}$
a	H^*	$H^* - \{-a^2\}$	$H - \{\pm 1\}$	$H^* - \{a\}$	$\{a\}$	$\{1, a^2, a^3\}$	P	$\{1, a, a^3\}$	P
a^2	$H^* - \{-a^3\}$	$H - \{\pm a\}$	$H^* - \{a^2\}$	H^*	$\{a^2\}$	P	$\{1, a, a^3\}$	P	$\{1, a, a^2\}$
a^3	$H - \{\pm a^2\}$	$H^* - \{a^3\}$	H^*	$H^* - \{-1\}$	$\{a^3\}$	$\{a, a^2, a^3\}$	P	$\{1, a, a^2\}$	P

TABLE 25. The table of hyperaddition for the tube $(15, 13, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a, -1\}$	$-P$	$-P$	$-P$	$\{-a^3\}$	$H^* - \{-1\}$	H^*	$H^* - \{a^3\}$	H
$-a^2$	$-P$	$\{-a^3, -a^2, -1\}$	$-P$	$-P$	$\{-a^2\}$	H^*	$H^* - \{a^2\}$	H	$H^* - \{-a^3\}$
$-a$	$-P$	$-P$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a\}$	$H^* - \{a\}$	H	$H^* - \{-a^2\}$	H^*
-1	$-P$	$-P$	$-P$	$\{-a^2, -a, -1\}$	$\{-1\}$	H	$H^* - \{-a\}$	H^*	$H^* - \{1\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-1\}$	H^*	$H^* - \{a\}$	H	$\{1\}$	$\{1, a, a^2\}$	P	P	P
a	H^*	$H^* - \{a^2\}$	H	$H^* - \{-a\}$	$\{a\}$	P	$\{a, a^2, a^3\}$	P	P
a^2	$H^* - \{a^3\}$	H	$H^* - \{-a^2\}$	H^*	$\{a^2\}$	P	P	$\{1, a^2, a^3\}$	P
a^3	H	$H^* - \{-a^3\}$	H^*	$H^* - \{1\}$	$\{a^3\}$	P	P	P	$\{1, a, a^3\}$

TABLE 26. The table of hyperaddition for the tube $(7, 15, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^3, -a^2, -1\}$	$-P$	$-P$	$-P$	$\{-a^3\}$	H^*	$H^* - \{-a, a^3\}$	H^*	H
$-a^2$	$-P$	$\{-a^3, -a^2, -a\}$	$-P$	$-P$	$\{-a^2\}$	$H^* - \{-1, a^2\}$	H^*	H	H^*
$-a$	$-P$	$-P$	$\{-a^2, -a, -1\}$	$-P$	$\{-a\}$	H^*	H	H^*	$H^* - \{-a^3, a\}$
-1	$-P$	$-P$	$-P$	$\{-a^3, -a, -1\}$	$\{-1\}$	H	H^*	$H^* - \{-a^2, 1\}$	H^*
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	H^*	$H^* - \{-1, a^2\}$	H^*	H	$\{1\}$	$\{1, a, a^3\}$	P	P	P
a	$H^* - \{-a, a^3\}$	H^*	H	H^*	$\{a\}$	P	$\{1, a, a^2\}$	P	P
a^2	H^*	H	H^*	$H^* - \{-a^2, 1\}$	$\{a^2\}$	P	P	$\{a, a^2, a^3\}$	P
a^3	H	H^*	$H^* - \{-a^3, a\}$	H^*	$\{a^3\}$	P	P	P	$\{1, a^2, a^3\}$

TABLE 27. The table of hyperaddition for the tube $(11, 15, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$-P$	$-P$	$-P$	$-P$	$\{-a^3\}$	H^*	H^*	H^*	H
$-a^2$	$-P$	$-P$	$-P$	$-P$	$\{-a^2\}$	H^*	H^*	H	H^*
$-a$	$-P$	$-P$	$-P$	$-P$	$\{-a\}$	H^*	H	H^*	H^*
-1	$-P$	$-P$	$-P$	$-P$	$\{-1\}$	H	H^*	H^*	H^*
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	H^*	H^*	H^*	H	$\{1\}$	P	P	P	P
a	H^*	H^*	H	H^*	$\{a\}$	P	P	P	P
a^2	H^*	H	H^*	H^*	$\{a^2\}$	P	P	P	P
a^3	H	H^*	H^*	H^*	$\{a^3\}$	P	P	P	P

TABLE 28. The table of hyperaddition for the tube $(15, 15, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2\}$	$\{-a^3, -a\}$	$\{-a^3, -1\}$	$\{-a^3\}$	$\{\pm 1, \pm a^3\}$	$\{\pm a, \pm a^3\}$	$\{\pm a^2, \pm a^3\}$	$H - \{\pm a^3\}$
$-a^2$	$\{-a^3, -a^2\}$	$\{-a^3, -a, -1\}$	$\{-a^2, -a\}$	$\{-a^2, -1\}$	$\{-a^2\}$	$\{\pm 1, \pm a^2\}$	$\{\pm a, \pm a^2\}$	$H - \{\pm a^2\}$	$\{\pm a^2, \pm a^3\}$
$-a$	$\{-a^3, -a\}$	$\{-a^2, -a\}$	$\{-a^3, -a^2, -1\}$	$\{-a, -1\}$	$\{-a\}$	$\{\pm 1, \pm a\}$	$H - \{\pm a\}$	$\{\pm a, \pm a^2\}$	$\{\pm a, \pm a^3\}$
-1	$\{-a^3, -1\}$	$\{-a^2, -1\}$	$\{-a, -1\}$	$\{-a^3, -a^2, -a\}$	$\{-1\}$	$H - \{\pm 1\}$	$\{\pm 1, \pm a\}$	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a^3\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$\{\pm 1, \pm a^3\}$	$\{\pm 1, \pm a^2\}$	$\{\pm 1, \pm a\}$	$H - \{\pm 1\}$	$\{1\}$	$\{a, a^2, a^3\}$	$\{1, a\}$	$\{1, a^2\}$	$\{1, a^3\}$
a	$\{\pm a, \pm a^3\}$	$\{\pm a, \pm a^2\}$	$H - \{\pm a\}$	$\{\pm 1, \pm a\}$	$\{a\}$	$\{1, a\}$	$\{1, a^2, a^3\}$	$\{a, a^2\}$	$\{a, a^3\}$
a^2	$\{\pm a^2, \pm a^3\}$	$H - \{\pm a^2\}$	$\{\pm a, \pm a^2\}$	$\{\pm 1, \pm a^2\}$	$\{a^2\}$	$\{1, a^2\}$	$\{a, a^2\}$	$\{1, a, a^3\}$	$\{a^2, a^3\}$
a^3	$H - \{\pm a^3\}$	$\{\pm a^2, \pm a^3\}$	$\{\pm a, \pm a^3\}$	$\{\pm 1, \pm a^3\}$	$\{a^3\}$	$\{1, a^3\}$	$\{a, a^3\}$	$\{a^2, a^3\}$	$\{1, a, a^2\}$

 TABLE 29. The table of hyperaddition for the tube $(14, 3, 5)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a^3, -a, -1\}$	$\{-a^3\}$	$H^* - \{-a\}$	$H^* - \{-1, a^2\}$	$H^* - \{1\}$	$H - \{\pm a^3\}$
$-a^2$	$\{-a^3, -a^2, -1\}$	$\{-a^3, -a, -1\}$	$\{-a^3, -a^2, -a\}$	$-P$	$\{-a^2\}$	$H^* - \{-a^3, a\}$	$H^* - \{a^3\}$	$H - \{\pm a^2\}$	$H^* - \{-1\}$
$-a$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-a^3, -a^2, -1\}$	$\{-a^2, -a, -1\}$	$\{-a\}$	$H^* - \{a^2\}$	$H^* - \{\pm a\}$	$H^* - \{-a^3\}$	$H^* - \{-a^2, 1\}$
-1	$\{-a^3, -a, -1\}$	$-P$	$\{-a^2, -a, -1\}$	$\{-a^3, -a^2, -a\}$	$\{-1\}$	$H - \{\pm 1\}$	$H^* - \{-a^2\}$	$H^* - \{-a, a^3\}$	$H^* - \{a\}$
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	$H^* - \{-a\}$	$H^* - \{-a^3, a\}$	$H^* - \{a^2\}$	$H - \{\pm 1\}$	$\{1\}$	$\{a, a^2, a^3\}$	$\{1, a, a^2\}$	P	$\{1, a, a^3\}$
a	$H^* - \{-1, a^2\}$	$H^* - \{a^3\}$	$H - \{\pm a\}$	$H^* - \{-a^2\}$	$\{a\}$	$\{1, a, a^2\}$	$\{1, a^2, a^3\}$	$\{a, a^2, a^3\}$	P
a^2	$H^* - \{1\}$	$H - \{\pm a^2\}$	$H^* - \{-a^3\}$	$H^* - \{-a, a^3\}$	$\{a^2\}$	P	$\{a, a^2, a^3\}$	$\{1, a, a^3\}$	$\{1, a^2, a^3\}$
a^3	$H - \{\pm a^3\}$	$H^* - \{-1\}$	$H^* - \{-a^2, 1\}$	$H^* - \{a\}$	$\{a^3\}$	$\{1, a, a^3\}$	P	$\{1, a^2, a^3\}$	$\{1, a, a^2\}$

 TABLE 30. The table of hyperaddition for the tube $(14, 7, 15)$.

+	$-a^3$	$-a^2$	$-a$	-1	0	1	a	a^2	a^3
$-a^3$	$\{-a^2, -a, -1\}$	$-P$	$-P$	$-P$	$\{-a^3\}$	H^*	H^*	H^*	$H - \{\pm a^3\}$
$-a^2$	$-P$	$\{-a^3, -a, -1\}$	$-P$	$-P$	$\{-a^2\}$	H^*	H^*	$H - \{\pm a^2\}$	H^*
$-a$	$-P$	$-P$	$\{-a^3, -a^2, -1\}$	$-P$	$\{-a\}$	H^*	$H - \{\pm a\}$	H^*	H^*
-1	$-P$	$-P$	$-P$	$\{-a^3, -a^2, -a\}$	$\{-1\}$	$H - \{\pm 1\}$	H^*	H^*	H^*
0	$\{-a^3\}$	$\{-a^2\}$	$\{-a\}$	$\{-1\}$	$\{0\}$	$\{1\}$	$\{a\}$	$\{a^2\}$	$\{a^3\}$
1	H^*	H^*	H^*	$H - \{\pm 1\}$	$\{1\}$	$\{a, a^2, a^3\}$	P	P	P
a	H^*	H^*	$H - \{\pm a\}$	H^*	$\{a\}$	P	$\{1, a^2, a^3\}$	P	P
a^2	H^*	$H - \{\pm a^2\}$	H^*	H^*	$\{a^2\}$	P	P	$\{1, a, a^3\}$	P
a^3	$H - \{\pm a^3\}$	H^*	H^*	H^*	$\{a^3\}$	P	P	P	$\{1, a, a^2\}$

 TABLE 31. The table of hyperaddition for the tube $(14, 15, 15)$.

6. THE CASE $N = 5$

The hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1, s_2)$. We have checked exactly 29791 cases, of which 2015 are hyperfields. This is 6.76379% of all cases, while non-isomorphic hyperfields are 521. This is 1.74885% of the total. The list of all non-isomorphic real hyperfields of order 11 with the cyclic positive cone of order $N = 5$ can be found in the files

- Hyperfields of C-char 1, of order 11, with the cyclic positive cone of order 5,
- Hyperfields of C-char 2, of order 11, with the cyclic positive cone of order 5.

There are 380 non-isomorphic real hyperfields of C-characteristic 1, and 141 non-isomorphic real hyperfields of C-characteristic 2. There are 491 isomorphism classes of order 4, and 21 isomorphism classes of order 2, and 9 classes of order 1.

7. THE CASE $N = 6$

The hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1, s_2, s_3)$. We have checked exactly 15752961 cases, of which 49321 are hyperfields. This is 0.31309% of all cases, while non-isomorphic hyperfields are 24750. This is 0.157113% of the total. The list of all non-isomorphic real hyperfields of order 13 with the cyclic positive cone of order $N = 6$ can be found in the files

- Hyperfields of C-char 1, of order 13, with the cyclic positive cone of order 6,

- Hyperfields of C-char 2, of order 13, with the cyclic positive cone of order 6.

There are 17915 non-isomorphic real hyperfields of C-characteristic 1, and 6835 non-isomorphic real hyperfields of C-characteristic 2. There are 24571 isomorphism classes of order 2, and 179 isomorphism classes of order 1.

8. THE CASE $N = 7$

The hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1, s_2, s_3)$. The list of all non-isomorphic real hyperfields of C-characteristic 3 of order 15 with the cyclic positive cone of order $N = 7$ can be found in the file

- Hyperfields of C-char 1, of order 15, with the cyclic positive cone of order 7,
- Hyperfields of C-char 2, of order 15, with the cyclic positive cone of order 7,
- Hyperfields of C-char 3, of order 15, with the cyclic positive cone of order 7.

There are 981522 non-isomorphic real hyperfields of C-characteristic 1, and 51022 non-isomorphic real hyperfields of C-characteristic 2, and 76 non-isomorphic real hyperfields of C-characteristic 3.

9. THE CASE $N = 8$

The hyperfield $\mathbb{H}(\underline{s})$ is given by the pair $\underline{s} = (s_0, s_1, s_2, s_3, s_4)$. The list of all non-isomorphic real hyperfields of C-characteristic 3 of order 17 with the cyclic positive cone of order $N = 8$ can be found in the file

- Hyperfields of C-char 3, of order 17, with the cyclic positive cone of order 8.

There are 190 non-isomorphic real hyperfields of C-characteristic 3.